Generalized power-spectrum Larmor formula for an extended charged particle embedded in a harmonic oscillator

Edwin A. Marengo and Mohamed R. Khodja
Department of Electrical and Computer Engineering, Northeastern University, Boston, Massachusetts 02115, USA
(Received 14 June 2006; published 15 September 2006)

The nonrelativistic Larmor radiation formula, giving the power radiated by an accelerated charged point particle, is generalized for a spatially extended particle in the context of the classical charged harmonic oscillator. The particle is modeled as a spherically symmetric rigid charge distribution that possesses both translational and spinning degrees of freedom. The power spectrum obtained exhibits a structure that depends on the form factor of the particle, but reduces, in the limit of an infinitesimally small particle and for the charge distributions considered, to Larmor's familiar result. It is found that for finite-duration small-enough accelerations as well as perpetual uniform accelerations the power spectrum of the spatially extended particle reduces to that of a point particle. It is also found that when the acceleration is violent or the size parameter of the particle is very large compared to the wavelength of the emitted radiation the power spectrum is highly suppressed. Possible applications are discussed.

DOI: 10.1103/PhysRevE.74.036611 PACS number(s): 03.50.De, 41.60.−m, 41.20.−q

I. INTRODUCTION

The idea of the incompatibility of the notion of charged structureless particles with classical electrodynamics has now grown into a consensus [1–4]. The incompatibility can be understood physically since classical physics starts to break down at length scales of the order of the Compton wavelength $\lambda_C \equiv h/mc$ (where h is Planck's constant, c is the speed of light in vacuum, and m is the mass of the particle in question.) At that level, and below for that matter, a quantum mechanical treatment becomes necessary. The problem of an extended charged particle has been investigated mainly from the standpoint of dynamics, that is, from the standpoint of the equation of motion. The reason behind this focus on the search for an equation of motion was to free the theory from the many pathologies that have plagued it (runaway solutions, acausal behavior, etc.) [1–6]. Another question of fundamental interest, which motivates our presentation, is how will endowing a particle with structure affect its radiative properties.

The instantaneous electromagnetic power radiated by a point particle of charge q moving with acceleration $\dot{\mathbf{V}}$ is given, in the nonrelativistic limit, by Larmor's formula

$$P = \frac{2}{3}\frac{q^2}{c^3}|\dot{\mathbf{V}}|^2. \tag{1}$$

Ford and O'Connell [7] have derived an extended particle generalization of (1) giving the instantaneous power in terms of the external force acting on the particle. The application of this formula requires prior knowledge of the equation of motion of the particle. However, this is not always easy to achieve for an extended charge distribution because of the structure-dependent radiation-reaction effects that must be taken into account. A generalization of the formula giving the radiated power directly in terms of the geometry of the particle is desirable, especially in view of the many applications that can be envisioned for this formula (see discussion of results). This task will be the focus of this paper. In particular, we shall derive a *generalized Larmor formula* that gives the power spectrum radiated by a particle endowed with structure when it undergoes both rectilinear and spinning motions. We need to mention, however, that in all that follows we do not necessarily mean by particle an *elementary particle* such as the electron. The so-called particle could well be a large polarized molecule or a nanoparticle and it is, in fact, in this spirit that the spinning motion of the particle is included in this analysis. There have also been doubts about the limits of applicability of Larmor's formula [8–10] and we will also shed some light on this question.

The formula we are seeking will reduce in the limit of an infinitesimally small particle to the familiar Larmor formula (1), though, physically speaking, it is illegitimate to take the limit, as we mentioned above. The obtained power spectrum has a structure that we feel may be relevant in a number of applications. Particular attention is given here to the classical (in contrast to quantum mechanical) charged harmonic oscillator case, but we also discuss how the theory can be extended to more general accelerated motions. The existence of certain nonradiating frequencies which yield no radiation as well as of frequencies for which the radiation is locally maximized is discussed, along with potential applications to aspects of classical electron theory and the measurement of particle sizes. Applications of the derived formula to novel electromechanical nanoantennas are also discussed. All equations are in the Gaussian system of units.

II. GENERAL RADIATION FORMULATION

Our main goal in this paper is to derive a generalized Larmor formula for the temporal frequency spectrum of the power radiated by a particle endowed with structure when it undergoes both rectilinear and spinning motions. The particle is modeled as corresponding to a spherically symmetric nonrelativistically rigid charge distribution which in its instantaneous rest frame is assumed to have the following spatial dependence:

$$\rho(|\mathbf{r}-\mathbf{R}(t)|)=qf[|\mathbf{r}-\mathbf{R}(t)|] \tag{2}$$

where q is the total charge of the particle, $\mathbf{R}(t)$ is its center of charge at time t, and f is the form factor subject to the normalization condition

$$\int f(r_0)d\mathbf{r}_0=1, \tag{3}$$

where here and henceforth $\mathbf{r}_0\equiv\mathbf{r}-\mathbf{R}(t)$ and $r_0\equiv|\mathbf{r}_0|$ and where the integral is taken over all space.

Our starting point is Maxwell's equations in vacuum, in particular ([11], p. 781),

$$\nabla\times\mathbf{E}(\mathbf{r},t)+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)=0,$$

$$\nabla\times\mathbf{B}(\mathbf{r},t)-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t)=\frac{4\pi}{c}\mathbf{j}(\mathbf{r},t), \tag{4}$$

where $\mathbf{E}$ and $\mathbf{B}$ are, respectively, the electric field and the magnetic flux density generated by the current distribution $\mathbf{j}$. It is easy to show that in the framework provided by Eqs. (4) the electric and magnetic fields produced by the source $\mathbf{j}$ are given in the far zone by [[12,13], Eqs. (5.48)–(5.50)]

$$\mathbf{E}(\mathbf{r},t)\sim\frac{1}{r}\mathbf{F}_e(\hat{\mathbf{r}},t_{ret})\equiv\frac{1}{rc^2}\frac{\partial}{\partial t}\{\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times\hat{\mathbf{j}}(\hat{\mathbf{r}},t_{ret})]\},$$

$$\mathbf{B}(\mathbf{r},t)\sim\frac{1}{r}\mathbf{F}_m(\hat{\mathbf{r}},t_{ret})\equiv-\frac{1}{rc^2}\frac{\partial}{\partial t}[\hat{\mathbf{r}}\times\hat{\mathbf{j}}(\hat{\mathbf{r}},t_{ret})], \tag{5}$$

where $r\equiv|\mathbf{r}|$ and $\hat{\mathbf{r}}\equiv\mathbf{r}/r$, $t_{ret}=t-r/c$ denotes the retarded time, and where we have introduced the electric and magnetic far-field patterns $\mathbf{F}_e$ and $\mathbf{F}_m$, respectively, which are in turn defined in terms of the slant-stack transform $\hat{\mathbf{j}}$ [12,14] of the current distribution $\mathbf{j}$, which is given by

$$\hat{\mathbf{j}}(\hat{\mathbf{s}},t)=\int\mathbf{j}\left(\mathbf{r},t+\frac{\mathbf{r}\cdot\hat{\mathbf{s}}}{c}\right)d\mathbf{r}, \tag{6}$$

where $\hat{\mathbf{s}}$ is a unit vector $\in S^2$ (where S^2 denotes the unit sphere) and having spherical coordinates $(1,\alpha,\beta)$ where α is the polar angle and β is the azimuthal angle, so that in Cartesian coordinates $\hat{\mathbf{s}}=(\sin\alpha\cos\beta,\sin\alpha\sin\beta,\cos\alpha)$. It is not hard to show by taking the temporal Fourier transform of both sides of (6) and inverting back to the time domain the result that

$$\hat{\mathbf{j}}(\hat{\mathbf{s}},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega t}\tilde{\mathbf{j}}(\hat{\mathbf{s}},\omega)d\omega \tag{7}$$

where the quantity $\tilde{\mathbf{j}}$ is defined as

$$\tilde{\mathbf{j}}(\hat{\mathbf{s}},\omega)\equiv\int_{-\infty}^{\infty}e^{i\omega t}dt\int\mathbf{j}(\mathbf{r},t)e^{-i(\omega/c)\hat{\mathbf{s}}\cdot\mathbf{r}}d\mathbf{r}, \tag{8}$$

which is identified as being the spatiotemporal Fourier transform

$$\int_{-\infty}^{\infty}e^{i\omega t}dt\int\mathbf{j}(\mathbf{r},t)e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r} \tag{9}$$

of the current function $\mathbf{j}(\mathbf{r},t)$ evaluated at $(\mathbf{k},\omega)=(\frac{\omega}{c}\hat{\mathbf{s}},\omega)$. The quantity $\tilde{\mathbf{j}}(\hat{\mathbf{s}},\omega)$ is then defined by the spatiotemporal Fourier transform (9) of the current density $\mathbf{j}(\mathbf{r},t)$ evaluated on the surface of the hypercone defined in spatiotemporal Fourier space by the relation $k\equiv|\mathbf{k}|=\frac{\omega}{c}$, or, alternatively, for each frequency ω, by the spatial Fourier transform of the frequency ω-valued temporal Fourier transform of the source evaluated on the surface of the so-called Ewald sphere of radius $\frac{\omega}{c}$ and center at the origin $(k=0)$ in spatial Fourier space.

The instantaneous power $P(t)$ radiated by the current distribution $\mathbf{j}(\mathbf{r},t)$ can be shown from Eqs. (5), (6) and Poynting's theorem (see, for example, [11], Chap. 6) to be given by

$$\begin{aligned}P(t)&=-\frac{c}{4\pi}\int_{S^2}\hat{\mathbf{r}}\cdot[\mathbf{F}_e(\hat{\mathbf{r}},t_{ret})\\&\quad\times\mathbf{F}_m(\hat{\mathbf{r}},t_{ret})]d\hat{\mathbf{r}}=\frac{c}{4\pi}\int_{S^2}|\mathbf{F}_e(\hat{\mathbf{s}},t_{ret})|^2d\hat{\mathbf{s}}\\&=\frac{1}{4\pi c^3}\int_{S^2}\left|\frac{\partial}{\partial t}\{[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot)-1]\hat{\mathbf{j}}(\hat{\mathbf{s}},t_{ret})\}\right|^2d\hat{\mathbf{s}}\end{aligned} \tag{10}$$

where $\cdot$ denotes the inner product and where we have used the identity $\hat{\mathbf{s}}\times\hat{\mathbf{s}}\times\equiv\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot)-1$. The total energy W_{rad} radiated by this source is then given by

$$\begin{aligned}W_{rad}&=\int_{-\infty}^{\infty}P(t)dt\\&=\frac{1}{4\pi c^3}\int_{-\infty}^{\infty}dt\int_{S^2}\left|\frac{\partial}{\partial t}\{[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot)-1]\hat{\mathbf{j}}(\hat{\mathbf{s}},t_{ret})\}\right|^2d\hat{\mathbf{s}}.\end{aligned} \tag{11}$$

For the case of charged harmonic oscillator motion of period T which will occupy us in the following it is more useful to consider the average power

$$P=\frac{1}{T}\int_T P(t)dt=\sum_\omega P_\omega, \tag{12}$$

where the integral is taken over any T duration interval and where P_ω is the individual contribution to the power spectrum of the frequency ω component of the associated Fourier series representation of the field [cf. Eq. (28)] so that (12) accounts for Parseval's relation for power signals [see, for example, [15], Eq. (4.1.14)].

III. POWER SPECTRUM OF AN EXTENDED CLASSICAL CHARGED HARMONIC OSCILLATOR

We wish to apply the above formulation to the current density of a spinning charged particle embedded to a harmonic oscillator. When in motion, the relevant charge distribution has current density [4]

$$\mathbf{j}(\mathbf{r},t) = \rho(r_0)[\mathbf{V}(t) + \Omega(t)\times\mathbf{r}_0] \tag{13}$$

where the charge density ρ is given by (2), $\mathbf{V}(t)\equiv\dot{\mathbf{R}}(t)$ is the particle's velocity (i.e., the rectilinear velocity of its center of charge), and $\mathbf{\Omega}(t)$ is the angular velocity of the particle in its rest frame. The motion of the particle's center of charge is assumed to be that of a harmonic oscillator, in particular,

$$\mathbf{R}(t) = Ae^{-i\omega_l t}\hat{\mathbf{z}} \tag{14}$$

where A represents the amplitude of the rectilinear oscillations, the z-axis unit vector $\hat{\mathbf{z}}$ is taken without loss of generality to define the direction of linear motion, and ω_l is the oscillator's angular frequency. Thus the velocity is

$$\mathbf{V}(t) = \dot{\mathbf{R}}(t) = -i\omega_l Ae^{-i\omega_l t}\hat{\mathbf{z}}. \tag{15}$$

Similarly, the rotational motion of the particle is also assumed to be that of a harmonic oscillator and, in particular, the angular velocity of the particle is taken to be of the form

$$\mathbf{\Omega}(t) = \Omega e^{-i\omega_s t}\hat{\mathbf{z}} \tag{16}$$

where ω_s is the oscillator's angular frequency and Ω is the magnitude of the spinning whose rotation axis is taken to coincide with that of the above-defined linear motion. The combination of linear motion in the direction defined by the unit vector $\hat{\mathbf{z}}$ and spinning with respect to the same direction thus contemplates the radiation of both electric and magnetic multipole moments, and for nanoantenna applications it can correspond to the combination of electric and magnetic dipoles in the same device. Generalization of the subsequent developments to more general linear and rotational motions, i.e., to more degrees of freedom, follows obvious lines, but we shall not dwell on this here.

For the current density expressed in (13) the quantity $\tilde{\mathbf{j}}$ in (8) is given by

$$\tilde{\mathbf{j}}(\hat{\mathbf{s}},\omega) = q\int_{-\infty}^{\infty} e^{i\omega t}dt \int [-i\hat{\mathbf{z}}A\omega_l e^{-i\omega_l t} + \Omega e^{-i\omega_s t}\hat{\mathbf{z}}\mathbf{r}_0]f(r_0)e^{-i(\omega/c)\hat{\mathbf{s}}\cdot\mathbf{r}}d\mathbf{r}, \tag{17}$$

i.e.,

$$\tilde{\mathbf{j}}(\hat{\mathbf{s}},\omega) = \tilde{\mathbf{j}}_l(\hat{\mathbf{s}},\omega) + \tilde{\mathbf{j}}_s(\hat{\mathbf{s}},\omega) \tag{18}$$

where

$$\begin{aligned}\tilde{\mathbf{j}}_l(\hat{\mathbf{s}},\omega) &= -\hat{\mathbf{z}}iA\omega_l q\int_{-\infty}^{\infty} e^{i(\omega-\omega_l)t}e^{-i(\omega/c)\hat{\mathbf{s}}\cdot\mathbf{R}(t)} \times dt\int f(r_0)e^{-i(\omega/c)\hat{\mathbf{s}}\cdot\mathbf{r}_0}d\mathbf{r} \\ &= -\hat{\mathbf{z}}iA\omega_l q\int_{-\infty}^{\infty} e^{i(\omega-\omega_l)t}e^{-i(\omega/c)\hat{\mathbf{s}}\cdot\mathbf{R}(t)} \times dt\int f(r_0)e^{-i(\omega/c)\hat{\mathbf{s}}\cdot\mathbf{r}_0}d\mathbf{r}_0\end{aligned} \tag{19}$$

is the part of the current that is due to the rectilinear motion, and

$$\begin{aligned}\tilde{\mathbf{j}}_s(\hat{\mathbf{s}},\omega) &= q\Omega\int_{-\infty}^{\infty} e^{i(\omega-\omega_s)t}e^{-i(\omega/c)\hat{\mathbf{s}}\cdot\mathbf{R}(t)}dt\int f(r_0) \times(\hat{\mathbf{z}}\mathbf{r}_0)e^{-i(\omega/c)\hat{\mathbf{s}}\cdot\mathbf{r}_0}d\mathbf{r} \\ &= q\Omega\int_{-\infty}^{\infty} e^{i(\omega-\omega_s)t}e^{-i(\omega/c)\hat{\mathbf{s}}\cdot\mathbf{R}(t)}dt\int f(r_0) \times(\hat{\mathbf{z}}\times\mathbf{r}_0)e^{-i(\omega/c)\hat{\mathbf{s}}\cdot\mathbf{r}_0}d\mathbf{r}_0\end{aligned} \tag{20}$$

is the part of the current that is due to the spinning motion. In the manipulations associated with Eqs. (19) and (20) we used change of integration variables, the final results corresponding physically to evaluation of the spatial part of the integrals in the rest frame of the particle. Dropping the subscript 0 from the radial and angular components of $\mathbf{r}_0$ within the integrals in Eqs. (19) and (20), we conveniently rewrite these results as

$$\tilde{\mathbf{j}}_l(\hat{\mathbf{s}},\omega) = \int_{-\infty}^{\infty} e^{i(\omega-\omega_l)t}e^{-i(\omega/c)R(t)\cos\alpha}dt\left(-\hat{\mathbf{z}}iA\omega_l q\int_0^{\infty} r^2 f(r) \times dr\int_0^{2\pi}d\varphi'\int_{-1}^{+1}e^{-i(\omega/c)r\cos\theta'}d(\cos\theta')\right) \tag{21}$$

where $R(t)\equiv|\mathbf{R}(t)|=Ae^{-i\omega_l t}$ and the polar and azimuthal angles θ' and φ', respectively, used in the integration, are defined with respect to the unit vector $\hat{\mathbf{s}}$ so that $\hat{\mathbf{s}}\cdot\hat{\mathbf{r}}=\cos\theta'$, and

$$\tilde{\mathbf{j}}_s(\hat{\mathbf{s}},\omega) = q\Omega\int_{-\infty}^{\infty} e^{i(\omega-\omega_s)t}e^{-i(\omega/c)R(t)\cos\alpha}dt\int f(r) \times(\hat{\mathbf{z}}\times\mathbf{r})e^{-i(\omega/c)r\cos\theta'}d\mathbf{r}. \tag{22}$$

One can cast (22) in a more useful form if one notes that $\hat{\mathbf{z}}\times\mathbf{r}=\hat{\varphi}r\sin\theta$. Thus (22) becomes

$$\tilde{\mathbf{j}}_s(\hat{\mathbf{s}},\omega) = \int_{-\infty}^{\infty} e^{i(\omega-\omega_s)t}e^{-i(\omega/c)R(t)\cos\alpha}dt \times\left(q\Omega\int_0^{\infty} r^3 f(r)dr\int_0^{2\pi}d\varphi' \times\int_{-1}^{+1}(\hat{\mathbf{z}}\times\hat{\mathbf{r}})e^{-i(\omega/c)r\cos\theta'}d(\cos\theta')\right). \tag{23}$$

Equations (10), (12), (7), (18), (21), and (23) form the basis of our formulation of the problem. Both Eqs. (21) and (23) can be put in the general form

$$\tilde{\mathbf{j}}_\xi(\hat{\mathbf{s}},\omega) = \int_{-\infty}^{\infty}\mathbf{I}_\xi(\omega)e^{i(\omega-\omega_\xi)t}e^{-i(\omega/c)R(t)\cos\alpha}dt \tag{24}$$

where $\xi\equiv l$ or s, depending on whether the motion in question is a translation or a spinning and $\mathbf{I}_\xi(\omega)$ is the time-independent integral defined as

$$\mathbf{I}_\xi(\omega) \equiv \begin{cases} -\hat{\mathbf{z}} iA\omega_l q \int_0^\infty r^2 f(r) dr \int_0^{2\pi} d\varphi' \int_{-1}^{+1} e^{-i(\omega/c) r \cos\theta'} d(\cos\theta') & \text{for the rectilinear motion,} \\ q\Omega \int_0^\infty r^3 f(r) dr \int_0^{2\pi} d\varphi' \int_{-1}^{+1} (\hat{\mathbf{z}}\times\hat{\mathbf{r}}) e^{-i(\omega/c) r\cos\theta'} d(\cos\theta') & \text{for the spinning motion.} \end{cases} \tag{25}$$

The vector character of $\mathbf{I}_\xi(\omega)$ is emphasized for future convenience.

From Eq. (24) and the Taylor series expansion of the exponential function one obtains

$$\tilde{\mathbf{j}}_\xi(\hat{\mathbf{s}},\omega) = \sum_{n=0}^{\infty} \mathbf{C}_{\xi n}(\omega)\,\delta(\omega - \Gamma_{\xi n}) \tag{26}$$

where

$$\mathbf{C}_{\xi n}(\omega) \equiv \frac{1}{n!}\left(-i\frac{\omega A}{c}\cos\alpha\right)^n \mathbf{I}_\xi(\omega) \tag{27}$$

and $\Gamma_{\xi n} \equiv \omega_\xi + n\omega_l$. Substituting Eq. (26) into Eq. (7) and then substituting the result into Eq. (12) yields

$$\begin{aligned} P_\xi &= \frac{1}{4\pi c^3 T}\int_{S^2} d\hat{\mathbf{s}} \int_T \left|\sum_{n=0}^{\infty}[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1]\mathbf{C}_{\xi n}(\Gamma_{\xi n})\Gamma_{\xi n} e^{-i\Gamma_{\xi n} t}\right|^2 dt \\ &= \frac{1}{4\pi c^3}\int_{S^2} d\hat{\mathbf{s}} \sum_{n=0}^{\infty}\Gamma_{\xi n}^2 |[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1]\mathbf{C}_{\xi n}(\Gamma_{\xi n})|^2 \\ &= \sum_{n=0}^{\infty}\frac{\Gamma_{\xi n}^2}{4\pi c^3}\int_{S^2} |[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1]\mathbf{C}_{\xi n}(\Gamma_{\xi n})|^2 d\hat{\mathbf{s}} \end{aligned} \tag{28}$$

where the individual terms of the infinite series correspond to the P_ω's of Eq. (12).

Equation (28) gives, depending on the $\mathbf{C}_{\xi n}$'s, the power due to the individual type of motion under investigation. The total power P when both types of motion are present is given by (see the Appendix)

$$P = P_l + P_s. \tag{29}$$

IV. CANONICAL CHARGE DISTRIBUTIONS AND THEIR POWER SPECTRA

So far no specific forms have been assumed for the form factor f. We shall consider the following simple charge distributions which we find physically plausible and mathematically tractable:

(1) a shell of radius a for which

$$f(r_0) = \frac{1}{4\pi a^2}\delta(a - r_0), \tag{30}$$

where δ is the Dirac delta function;

(2) a uniform sphere of radius a for which

$$f(r_0) = \frac{3}{4\pi a^3}\Theta(a - r_0), \tag{31}$$

where Θ is the Heaviside unit step function;

(3) a "naive" negative exponential with size parameter a for which

$$f(r_0) = \frac{1}{8\pi a^3}e^{-r_0/a}; \tag{32}$$

(4) and a more elaborate Yukawa-type charge distribution with size parameter a [16] for which

$$f(r_0) = \frac{1}{4\pi a^2}\frac{e^{-r_0/a\sqrt{2}}}{r_0}\sin\left(\frac{r_0}{a\sqrt{2}}\right). \tag{33}$$

Note that all the above charge distributions satisfy the normalization condition (3).

The first two charge distributions, namely, (30) and (31), have compact support, and are of interest for both the macroscopic and elemental particle points of view (in fact, the first one has been the focus of past investigations in this area), while the other two, namely, (32) and (33), are noncompactly supported, yet rapidly decaying distributions, familiar to particle models. Incidentally, one may note that for the latter two distributions we opted for distributions that have no "core" instead of something that varies as $e^{-Cr_0^n}$ (where C is a positive constant and $n=2,3,4,\ldots$), and displays a "hard core" which becomes harder as n increases. When we made this choice we had in mind the fact that deep inelastic scattering experiments in high-energy physics as well as precision magnetic-moment measurements indicate that the electron has no structure down to distances $\sim 10^{-19}$ m; this is several orders of magnitude smaller than the Compton wavelength of the electron ($\sim 10^{-12}$ m) and the so-called "classical electron radius" ($\sim 10^{-15}$ m). (For a concise review of the experimental limits on the characteristic sizes of elementary particles see, for instance, [17].) We may also note that we deliberately avoided the "naive" Yukawa distribution $r_0^{-1}e^{-Cr_0}$ because it exhibits a singular behavior at the origin which, in our opinion, does not distinguish it much from the singular point-particle "distribution."

A. Evaluation of the power spectrum due to the rectilinear motion of the particle

For the rectilinear motion it can easily be shown that (28) takes on the explicit form

$$P_l = \frac{8\pi^2}{c^3} q^2 (\omega_l^2 A)^2 \sum_{n=0}^{\infty} \left(\frac{n+1}{n!}\right)^2 \left| \int_0^\infty r^2 f(r) \frac{\sin\left[\frac{(n+1)\omega_l}{c} r\right]}{\frac{(n+1)\omega_l}{c} r} dr \right|^2 \times \int_0^\pi \left[\frac{(n+1)\omega_l A}{c} \cos\alpha\right]^{2n} \sin^3\alpha \, d\alpha. \quad (34)$$

1. Rigid spherical shell

In this case

$$\left| \int_0^\infty r^2 f(r) \frac{\sin \frac{(n+1)\omega_l}{c} r}{\frac{(n+1)\omega_l}{c} r} dr \right|^2 = \left| \int_0^\infty r^2 \frac{\delta(r-a)}{4\pi a^2} \frac{\sin\left[\frac{(n+1)\omega_l}{c} r\right]}{\frac{(n+1)\omega_l}{c} r} dr \right|^2 = \left\{ \frac{1}{4\pi} \frac{\sin\left[\frac{(n+1)\omega_l}{c} a\right]}{\frac{(n+1)\omega_l}{c} a} \right\}^2. \quad (35)$$

Substituting Eq. (35) into Eq. (34) yields

$$P_l^{(S)} = \frac{8\pi^2}{c^3} q^2 (\omega_l^2 A)^2 \sum_{n=0}^{\infty} \left(\frac{n+1}{n!}\right)^2 \times \left\{ \frac{1}{4\pi} \frac{\sin\left[\frac{(n+1)\omega_l}{c} a\right]}{\frac{(n+1)\omega_l}{c} a} \right\}^2 \times \int_0^\pi \left[\frac{(n+1)\omega_l A}{c} \cos\alpha\right]^{2n} \sin^3\alpha \, d\alpha. \quad (36)$$

Expressing the leading three terms of the power spectrum (i.e., $n=0,1,2$) this gives

$$P_l^{(S)} = \frac{8\pi^2}{c^3} q^2 (\omega_l^2 A)^2 \left(\frac{1}{4\pi}\right)^2 \int_0^\pi d\alpha \sin^3\alpha \left\{ \left[\omega_l \frac{\sin\left(\frac{\omega_l}{c} a\right)}{\left(\frac{\omega_l}{c} a\right)} \right]^2 + \left[2\omega_l \frac{\sin\left(\frac{2\omega_l}{c} a\right)}{\left(\frac{2\omega_l}{c} a\right)} \left(\frac{2\omega_l A}{c} \cos\alpha\right) \right]^2 + \left[\frac{9}{2}\omega_l \frac{\sin\left(\frac{3\omega_l}{c} a\right)}{\left(\frac{3\omega_l}{c} a\right)} \left(\frac{3\omega_l A}{c} \cos\alpha\right)^2 \right]^2 + \cdots \right\}. \quad (37)$$

At this point, we note that we are working in the nonrelativistic limit. Thus one can keep only the zeroth-order term and neglect all the other terms as they all contain $\left(\frac{\omega_l A}{c}\right)^2 = \left(\frac{|\mathbf{V}|}{c}\right)^2 \equiv \left(\frac{V}{c}\right)^2$. Hence, under the requirement that

$$\left(\frac{\omega_l A}{c}\right)^2 = \left(\frac{V}{c}\right)^2 \ll 1 \quad (38)$$

the only term in Eq. (37) that survives is the first one, and one ends up with

$$P_l^{(S)} = \frac{2q^2}{3c^3} (A\omega_l^2)^2 \left[\frac{\sin\left(\frac{\omega_l}{c} a\right)}{\left(\frac{\omega_l}{c} a\right)} \right]^2. \quad (39)$$

Note that (39) is Larmor's familiar formula (1) corrected by $\left[\sin\left(\frac{\omega_l}{c}a\right)/\left(\frac{\omega_l}{c}a\right)\right]^2$ (the modulus of the acceleration being $A\omega_l^2$).

2. Uniform sphere

In a fashion similar to that of the previous case, one can show that in the nonrelativistic limit (38) the power spectrum reduces to

$$P_l^{(U)} = \frac{2q^2}{3c^3} (A\omega_l^2)^2 \left\{ \frac{9}{\left(\frac{\omega_l}{c} a\right)^4} \left[\cos\left(\frac{\omega_l}{c} a\right) - \frac{\sin\left(\frac{\omega_l}{c} a\right)}{\left(\frac{\omega_l}{c} a\right)} \right]^2 \right\}. \quad (40)$$

3. Negative exponential charge distribution

In this case the power spectrum reduces to

$$P_l^{(E)} = \frac{2q^2}{3c^3} (A\omega_l^2)^2 \frac{1}{\left[1 + \left(\frac{\omega_l}{c} a\right)^2\right]^4}. \quad (41)$$

4. Yukawa-type charge distribution

The power spectrum is in this case

$$P_l^{(Y)} = \frac{2q^2}{3c^3}(A\omega_l^2)^2 \frac{1}{\left[1+\left(\frac{\omega_l}{c}a\right)^4\right]^2} \tag{42}$$

which is very similar to (41).

B. Evaluation of the power spectrum due to the spinning of the particle

For the spinning motion (28) takes on the explicit form

$$P_s = \frac{8\pi^2}{c^3}q^2\Omega^2\sum_{n=0}^{\infty}\left(\frac{\omega_s+n\omega_l}{n!}\right)^2\left|\int_0^\infty r^3 f(r)\frac{1}{\left(\frac{\omega_s+n\omega_l}{c}r\right)^2}\right.$$
$$\times\left[\left(\frac{\omega_s+n\omega_l}{c}r\right)\cos\left(\frac{\omega_s+n\omega_l}{c}r\right)\right.$$
$$\times-\left.\sin\left(\frac{\omega_s+n\omega_l}{c}r\right)\right]dr\left|^2\int_0^\pi\left[\frac{(\omega_s+n\omega_l)A}{c}\right.\right.$$
$$\times\cos\alpha\Bigg]^{2n}\sin^3\alpha\, d\alpha. \tag{43}$$

The details of the manipulations leading to this result are given in the Appendix.

1. Rigid spherical shell

Following the same procedure used before we arrive at

$$P_s^{(S)} = \frac{8\pi^2}{c^3}q^2\Omega^2\left(\frac{a}{4\pi}\right)^2\sum_{n=0}^{\infty}\left(\frac{\omega_s+n\omega_l}{n!}\right)^2\frac{1}{\left(\frac{\omega_s+n\omega_l}{c}a\right)^4}$$
$$\times\left[\left(\frac{\omega_s+n\omega_l}{c}a\right)\cos\left(\frac{\omega_s+n\omega_l}{c}a\right)\right.$$
$$-\left.\sin\left(\frac{\omega_s+n\omega_l}{c}a\right)\right]^2\int_0^\pi\left[\frac{(\omega_s+n\omega_l)A}{c}\right.$$
$$\times\cos\alpha\Bigg]^{2n}\sin^3\alpha\, d\alpha. \tag{44}$$

Hence,

$$P_s^{(S)} = \frac{8\pi^2}{c^3}q^2\Omega^2\left(\frac{a}{4\pi}\right)^2\int_0^\pi \sin^3\alpha\, d\alpha\left[\omega_s^2\frac{1}{\left(\frac{\omega_s}{c}a\right)^4}\left[\left(\frac{\omega_s}{c}a\right)\cos\left(\frac{\omega_s}{c}a\right)-\sin\left(\frac{\omega_s}{c}a\right)\right]^2\right.$$
$$+\left(\frac{1}{\left(\frac{\omega_s+\omega_l}{c}a\right)^2}\left[\left(\frac{\omega_s+\omega_l}{c}a\right)\cos\left(\frac{\omega_s+\omega_l}{c}a\right)-\sin\left(\frac{\omega_s+\omega_l}{c}a\right)\right]\frac{(\omega_s+\omega_l)A}{c}\cos\alpha\right)^2$$
$$+\frac{1}{2}\left(\frac{1}{\left(\frac{\omega_s+2\omega_l}{c}a\right)^2}\left[\left(\frac{\omega_s+2\omega_l}{c}a\right)\cos\left(\frac{\omega_s+2\omega_l}{c}a\right)\right.\right.$$
$$\left.\left.-\left.\sin\left(\frac{\omega_s+2\omega_l}{c}a\right)\right]\left(\frac{(\omega_s+2\omega_l)A}{c}\cos\alpha\right)^2\right)^2+\cdots\right] \tag{45}$$

which under condition (38) and the requirement that

$$\left(\frac{\omega_s A}{c}\right)^2 \ll 1 \tag{46}$$

reduces to

$$P_s^{(S)} = \frac{2}{3c}q^2\Omega^2\left[\cos\left(\frac{\omega_s}{c}a\right)-\frac{\sin\left(\frac{\omega_s}{c}a\right)}{\left(\frac{\omega_s}{c}a\right)}\right]^2. \tag{47}$$

The new condition (46) arises because of the form of the angular velocity adopted in this work [cf. Eq. (16)]. It can be understood as a requirement that

$$\omega_s \lesssim \omega_l^{(\max)} \tag{48}$$

where $\omega_l^{(\max)}$ stands for the upper limit on the rectilinear motion frequency which keeps the results within their domain of applicability [see (58) and the discussion leading to it].

2. Uniform sphere

Following the same procedure as before we find that, under conditions (38) and (48), the power spectrum reduces to

$$P_s^{(U)} = \frac{6}{c}q^2\Omega^2\frac{1}{(\omega_s a/c)^6}\left\{3\left(\frac{\omega_s}{c}a\right)\cos\left(\frac{\omega_s}{c}a\right) + \left[\left(\frac{\omega_s}{c}a\right)^2 - 3\right]\sin\left(\frac{\omega_s}{c}a\right)\right\}^2. \tag{49}$$

3. Negative exponential

In this case

$$P_s^{(E)} = \frac{32}{3c}q^2\Omega^2\frac{(\omega_s a/c)^4}{[1+(\omega_s a/c)^2]^6}. \tag{50}$$

4. Yukawa-type distribution

In this case

$$P_s^{(Y)} = \frac{32}{3c}q^2\Omega^2\frac{(\omega_s a/c)^8}{[1+(\omega_s a/c)^4]^4}. \tag{51}$$

Note that (51) is similar to (50).

V. DISCUSSION OF THE RESULTS

Equations (39)–(42) give the power radiated due to the rectilinear motion by the shell, the uniform sphere, the exponential, and the Yukawa-type charge distributions, respectively. To zeroth order (imposed by the requirement that the motion be nonrelativistic) the frequency of the emitted radiation is equal to the oscillation frequency ω_l. It appears that higher harmonics $n\omega_l$ become important only in the relativistic case, although a fully relativistic treatment would be necessary in order to confirm this observation. The fact that Eqs. (39) and (40) exhibit nonradiating modes is a known result [18]. What requires special attention, however, is that the behavior of these power spectra is not governed solely by the size of the particle but rather by the dimensionless parameter $\omega_l a/c$, and this gives rise to some interesting behavior. First of all, we note that for a given ω_l we recover Larmor's familiar result for a point particle (i.e., as $a \to 0$). Physically, this corresponds to $a \ll \lambda_{rad}$ where $\lambda_{rad} \equiv 2\pi c/\omega$ is the radiation wavelength. This is true even in the case of (40) for which $\frac{\omega_l}{c}a=0$ is a singular point. Second, for a given a, the limit $\omega_l \to 0$ (which is physically equivalent to $T \gg \tau$ where $T \equiv 2\pi/\omega$ is the period of oscillatory motion and $\tau \equiv a/c \sim$ the time it takes light to cross the particle) this limit reproduces the same results as the point particle limit. This is interesting because what this means is that a particle endowed with structure can have a power spectrum similar to that of a point particle provided that ω_l is small enough. We need to distinguish two cases here: (a) finite-amplitude motions and (b) motions in which the amplitude is allowed to go to infinity. Indeed, $A\omega_l^2 = |\dot{\mathbf{V}}|$ being the magnitude of the rectilinear acceleration, these two cases lead to two different situations. If the amplitude A is finite $A\omega_l^2$ will be very small and the above conclusion (the fact that the power spectrum of a sizeable particle will look like that of a point particle) corresponds to the case of the particle moving with a small acceleration. In other words, for small enough accelerations that last for a finite time the power spectrum of a particle with structure will look like the power spectrum of a point particle. If, on the other hand, A is allowed to have very large values, then one can imagine a process by which ω_l^2 is decreased at the same time A is being taken to infinity such that $A\omega_l^2$ remains finite, and takes on any value, provided that the validity condition (38) is respected. By doing so one recovers the case of the perpetual uniformly accelerated motion. This amounts to saying that for a sizable particle undergoing perpetual uniform acceleration the power spectrum looks exactly like that of a point particle. It has been claimed that no radiation is emitted by a point particle in the case of a perpetual motion and that only finite-duration accelerations cause the point particle to radiate at a frequency $\omega \sim 1/T$ where T is the duration of the acceleration [10]. Our results suggest otherwise; they suggest that (a) for finite-duration small-enough accelerations as well as perpetual uniform accelerations the power spectrum of a finite-size particle reduces to that of a point particle, and (b) for a point particle the emitted power does not vanish. Also, our results, though entirely nonrelativistic, seem to agree with the conclusion that for violent accelerations the radiation emitted is greatly suppressed in the relativistic regime [9]. Indeed, for large ω_l the control parameter $\omega_l a/c \gg 1$ (where a is fixed), and the radiated power given by Eqs. (39)–(42) is greatly suppressed [this too is subject to the validity condition (38)]. This is also true for very large size parameters a (i.e., $a \gg \lambda_{rad}$).

We need to clarify one issue at this point. The motion of a harmonic oscillator has the peculiar property that, to any order n, the derivatives of the position vector are never identically zero. Indeed, $d^n\mathbf{R}(t)/dt^n = (-i\omega_l)^n A e^{-i\omega_l t}\hat{\mathbf{z}}$. In particular this implies that the acceleration $\ddot{\mathbf{R}}(t)$ varies at every point in time. Thus a question arises as to whether under the limiting process described above the mere requirement that $A\omega_l^2$ be finite for large A and small ω_l will guarantee a uniform acceleration. Supposing that the limiting process yields $A\omega_l^2 \sim 1$ we deduce that $\omega_l^n \sim A^{-n/2}$. Consequently, $|d^n\mathbf{R}(t)/dt^n| \sim \omega_l^{n-2}\omega_l^2 A \sim A^{1-n/2}$. We clearly see that for $n \geq 3, |d^n\mathbf{R}(t)/dt^n| \sim A^{-m/2}$, where $m \in \mathbb{N}^*$. Hence, $|d^n\mathbf{R}(t)/dt^n| \underset{A\to\infty}{\to} 0$, i.e., the acceleration is indeed uniform in this case.

The correction factors appearing in Eqs. (39)–(42) all display maximum values in the limit $(\omega_l c/a) \to 0$. Hence, the maximum power is obtained in the point particle limit. In a sense, having a large particle is a way of preserving its electromagnetic energy. For instance, the power radiated by the spherical shell falls off rapidly as $\omega_l c/a$ increases; it is only about 5% of the power emitted by a point particle for a value of $\omega_l c/a$ as low as ~ 2.5. Obviously, the more extended the particle the greater is the interference between the fields emitted by its different parts (decoherence), whence the reduction of the radiated power.

Theoretically, the formula derived by Ford and O'Connell [7] for the instantaneous power radiated by an extended nonspinning particle should yield results similar to ours, provided that the equation of motion is known in advance. As an informative example we carry out an explicit calculation of the power $P_l^{(S)}$ for a shell of radius a. The Ford-O'Connell formula reads [7]

$$P(t) = \frac{2q^2}{3c^3}\left|\frac{\mathbf{F}_{ext}(t)}{m_{obs}}\right|^2, \quad (52)$$

where m_{obs} is the observed (physical or renormalized) mass of the particle and $\mathbf{F}_{ext}(t)$ the external force acting on it. Sommerfeld has shown that the equation of motion of a rigid spherical shell in the nonrelativistic limit is, to a good approximation, given by [3]

$$\mathbf{F}_{ext}(t) = (m_{obs} - m_{el}) \cdot \dot{\mathbf{V}}(t) - \frac{m_{el}}{2a}\left[\mathbf{V}\left(t - \frac{2a}{c}\right) - \mathbf{V}(t)\right] \quad (53)$$

where $m_{el} \equiv 2q^2/3c^2a$ is the electromagnetic mass. This equation is subject to the validity condition $m_{obs} > m_{el}$. For a harmonic oscillator Eqs. (53) and (52) lead to

$$P_l^{(S)}(t) = \frac{2q^2}{3c^3}(A\omega_l^2)^2\left|1 - \frac{m_{el}}{m_{obs}} - \frac{m_{el}}{m_{obs}}e^{i(\omega_l/c)a}\frac{\sin(\omega_l a/c)}{\omega_l a/c}\right|^2. \quad (54)$$

At this point we need to make an additional assumption in order to move forward. While still respecting the validity condition $m_{obs} > m_{el}$, one can always choose the size parameter (in this case the radius a) such that

$$\frac{m_{el}}{m_{obs}} \simeq 1. \quad (55)$$

Therefore, to a very good approximation

$$P_l^{(S)} = \frac{2q^2}{3c^3}(A\omega_l^2)^2\left[\frac{\sin(\omega_l a/c)}{(\omega_l a/c)}\right]^2. \quad (56)$$

This result is the same as (39). The fact that we had to use an additional assumption [namely, (55)] to arrive at (56) is probably due to the approximate nature of the equation of motion (53).

We now turn to the discussion of the power spectra due to the spinning motion. Equations (47), (49), (50), and (51) give the power radiated due to the spinning motion of the shell, sphere, exponential, and Yukawa-type distributions, respectively. We note the following general features of the power spectra.

(1) The frequency of the radiation is equal to ω_s. (As in the case of the rectilinear motion it appears that radiation with higher frequencies $\omega_s + n\omega_l$ becomes important only in the relativistic regime. But, as indicated above, the vindication of this observation will have to wait for a fully relativistic treatment of the problem.)

(2) $\lim_{\omega_s a/c \to 0} P_s = 0$ for all four charge distributions, as expected for point particles ($a \to 0$ while ω_s is fixed) or when the spinning takes place with constant angular velocity Ω ($\omega_s = 0$ while a is fixed).

(3) When $\omega_s a/c \gg 1$ the radiation is highly suppressed [except for (47); see Figs. 1 and 2].

It is also worth noting that in Eq. (47), the power spectrum $P_s^{(S)}$ exhibits nonradiating modes at the approximate values $\frac{\omega_s}{c}a \simeq 4.493, 7.725$, etc. The first and greatest maximum of $P_s^{(S)}$ occurs at $\frac{\omega_s}{c}a \simeq 2.744$, and at $\frac{\omega_s}{c}a \simeq 6.117, 9.317$, etc., it has maxima that for large $\frac{\omega_s}{c}a$ tend to the magnitude unity. Equation (49) too exhibits nonradiating modes at $\frac{\omega_s}{c}a \simeq 5.762, 9.092$, etc., and the first and greatest maximum of $P_s^{(U)}$ occurs at $\frac{\omega_s}{c}a \simeq 3.342$. The maximum of (50) occurs at $\frac{\omega_s}{c}a = 1/\sqrt{2}$ and the maximum of (51) occurs at $\frac{\omega_s}{c}a = 1$.

In addition to the validity conditions imposed by the nonrelativistic formulation of the problem, i.e., Eqs. (38) and (46), we need to impose another validity condition related to the fact that the frequency of emitted radiation ω is equal to the frequency of the motion ω_ξ where $\xi = l$ or s. To illustrate this let us consider the example of a particle undergoing rectilinear motion only. (A similar argument holds for a particle with spinning motion.) A typical energy of the oscillator is the kinetic energy of the particle which is $\sim mV^2$. In order for us to be able to overlook the quantum mechanical nature of radiation we must have

$$mV^2 \gg \hbar\omega \quad (57)$$

where ω is the angular frequency of the emitted radiation. Taking into account the fact that $\omega = \omega_l$, and invoking (38) we get

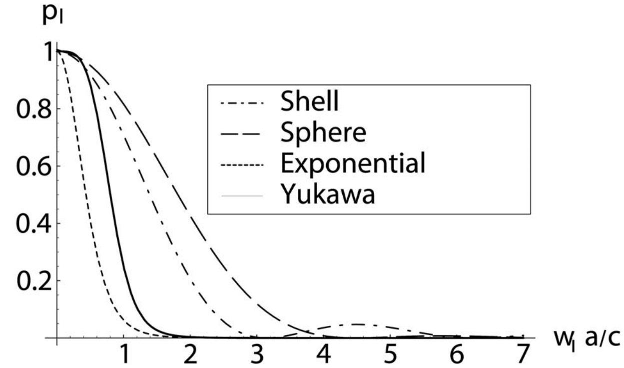

FIG. 1. Profiles of the power emitted due to the rectilinear motion. (The power is normalized with respect to the power emitted by a point particle.)

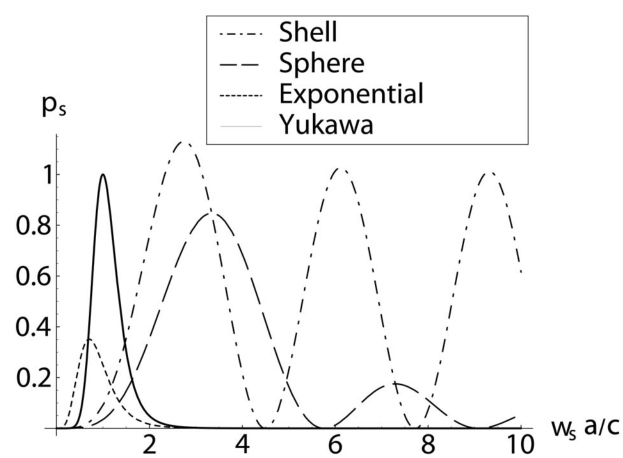

FIG. 2. Profiles of the power emitted due to the spinning motion. [The power is normalized with respect to $(2/3c)q^2\Omega^2$.]

$$\omega_l \ll \frac{mc^2}{\hbar} \sim 10^{47} m \tag{58}$$

where $mc^2/\hbar \equiv \omega_l^{(\max)}$ [cf. Eq. (48)].

For instance, even if mass m is as small as that of an electron, Eq. (58) is telling us that the quantum nature of radiation can safely be disregarded as long as the frequency of oscillation is much smaller than 10^{20} Hz$\equiv 10^{11}$ GHz. Condition (58) allows for a wide range of applicability of our results, especially for the engineering applications discussed below. (As a matter of fact the highest nanomechanical resonance frequency achieved in the laboratory is $\sim$1 GHz [19].)

In a similar fashion one can put an upper limit on the accelerations for which our results hold. Combining Eqs. (58) and (38) yields a magnitude of $\omega_l^2 A \ll mc^3/\hbar \sim 10^{58} m$. For an electron this upper limit takes on a value $\sim 10^{31}$ cm s^{-2}.

Our result is primarily meant as a generalization of Larmor's formula. Thus, in principle, it should be possible to use it as a substitute for Larmor's formula whenever the spatial extent of the emitting particle is relevant. At the fundamental level our result could be used to characterize the size of a radiating particle by means of parametric estimation, for instance. (Here we reiterate the fact that by "particle" we not necessarily mean an elementary particle such as the electron.) Plasma physics also is a domain where Larmor's formula is routinely used, though in its relativistic form, to calculate energy losses both in high-energy experiments and in astrophysics. A relativistic version of the result may give some additional insight. At a more practical level, our result could be used to calculate the power emitted by an electromechanical nanoantenna. It has already been suggested that nanoantennas be used as solutions to the so-called nanointerconnect problem [20]: how to connect the nanoworld to the macroscopic world. As noted in [20], all the nanodevices manufactured until now have been contacted by lithographically made electrodes. But the potential high-resolution circuitry made possible with the advent of nanotubes will not be achieved if each and every nanotube is to be contacted lithographically. One way of solving this problem is to use wireless interconnects. For instance, one could connect the interconnects to nanoantennas of different lengths (hence different resonant frequencies), easing at the same time the need for high-cost lithography [20]. It may be envisioned that electromechanical nanoantennas will play a similar role. Other proposed applications of nanoantennas, where the formula may be useful too, is in the area of sensing where nanoantennas would be coupled to chemical and biological nanosensors sensitive to their chemical environment without having to recur to lithography [20]. Of course the problem of the practical viability of the kind electromechanical nanoantennas described above has yet to be explored (inclusion of the effects of driving forces, damping, etc.).

Possible future followups of this work would be to devise either a covariant formulation of the problem or a quantum mechanical one. Of course, a quantum electrodynamic formulation valid for relativistic velocities and very high frequencies at the same time would be ideal.

VI. CONCLUSION

In summary we have generalized the nonrelativistic Larmor radiation formula, giving the power radiated by an accelerated charged point particle, for a spatially extended charged particle in the context of the classical charged harmonic oscillator. We modeled the particle as a spherically symmetric rigid charge distribution that possesses both translational and spinning degrees of freedom. The obtained formula has the convenience that it does not require the prior knowledge of the equation of motion but gives instead the power spectrum in terms of the form factor of the particle. We also carried out explicit calculations of the power spectrum for a selection of physically plausible and mathematically tractable charge distributions. The power spectrum obtained exhibits a structure that depends on the form factor of the particle (nonradiating modes, maxima, etc.), but reduces, in the limit of an infinitesimally small particle (and for the charge distributions considered), to Larmor's familiar result. Under the requirement that the motion be nonrelativistic we found that the radiation frequency is equal to the frequency of oscillation (for both the rectilinear motion and the spinning motion). We also found that the quantum nature of emitted radiation could be neglected for frequencies $\ll 10^{47} m$ (in Hz) where m is the mass of the particle. For finite-duration small-enough accelerations as well as perpetual uniform accelerations the power spectrum of the spatially extended particle reduces to that of a point particle. It is also found that when the acceleration is violent or the size parameter of the particle is very large compared to the wavelength of the emitted radiation this one is highly suppressed [the requirement that the motion be nonrelativistic and the radiation have no quantum mechanical effects on the oscillator stipulates that the magnitude of the rectilinear acceleration be $\ll 10^{58} m$ (in cm s^{-2}) where m is the mass of the particle]. Possible applications of our results both at the fundamental level and the practical level have been discussed.

ACKNOWLEDGMENTS

This work was supported by the Air Force Office of Scientific Research under Grant No. FA9550-06-01-0013, and is affiliated with CenSSIS, the Center for Subsurface Sensing and Imaging Systems, under the Engineering Research Centers Program of the National Science Foundation (Grant No. EEC-9986821).

APPENDIX

The goal of this appendix is twofold: to prove (a) that Eq. (28) yields Eq. (43), and (b) Eq. (29). The idea is to perform the calculations in a coordinate system that simplifies the exponential $\exp\left(-i\frac{\omega r}{c}\hat{\mathbf{s}}\cdot\hat{\mathbf{r}}\right)$ appearing in Eqs. (19) and (20), i.e., a system in which $\hat{\mathbf{s}}\cdot\hat{\mathbf{r}}=\cos\theta'$. This new coordinate system is a rotated system in which $\hat{\mathbf{s}}$ plays the role of the z axis. In other words it is the coordinate system obtained by letting $\hat{\mathbf{x}}\to\hat{\mathbf{x}}', \hat{\mathbf{y}}\to\hat{\mathbf{y}}'$, and $\hat{\mathbf{z}}\to\hat{\mathbf{z}}'\equiv\hat{\mathbf{s}}$ through a rotation of Euler angle α around an axis perpendicular to the plane formed by $\hat{\mathbf{s}}$ and $\hat{\mathbf{z}}$ and passing through the origin.

The triple vector product rule allows us to write

$$\hat{\mathbf{s}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{z}}) = (\hat{\mathbf{s}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{z}} + \hat{\mathbf{s}}(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}) - \hat{\mathbf{z}}(\hat{\mathbf{s}} \cdot \hat{\mathbf{r}})$$
$$= \sin\theta' \hat{\varphi}' \times \hat{\mathbf{z}} + \cos\theta \hat{\mathbf{s}} - \cos\theta' \hat{\mathbf{z}}. \quad \text{(A1)}$$

The cross product $\hat{\mathbf{s}} \times [\hat{\mathbf{s}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{z}})] = [\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1](\hat{\mathbf{z}} \times \hat{\mathbf{r}})$ is then given by

$$\hat{\mathbf{s}} \times [\hat{\mathbf{s}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{z}})] = \sin\theta' \hat{\mathbf{s}} \times (\hat{\varphi}' \times \hat{\mathbf{z}}) - \cos\theta' \hat{\mathbf{s}} \times \hat{\mathbf{z}}$$
$$= \sin\theta' \cos\alpha \hat{\varphi}' + \sin\alpha \cos\theta' \hat{\varphi}'' \quad \text{(A2)}$$

where $\hat{\varphi}' \equiv \cos\varphi' \hat{\mathbf{y}}' - \sin\varphi' \hat{\mathbf{x}}'$ and $\hat{\varphi}'' \equiv \cos\beta \hat{\mathbf{y}} - \sin\beta \hat{\mathbf{x}} \equiv \hat{\varphi}|_{\theta=\alpha,\varphi=\beta}$.

Proof that Eq. (28) can be cast into Eq. (43)

In order to prove that Eq. (28) becomes Eq. (43) we need to evaluate $[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot)-1]\mathbf{I}_\xi(\omega)$, which is essentially $[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot)-1] \times (\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = \hat{\mathbf{s}} \times [\hat{\mathbf{s}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{z}})]$ so that from Eq. (A2)

$$[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1](\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = \sin\theta' \cos\alpha \hat{\varphi}' + \sin\alpha \cos\theta' \hat{\varphi}''.$$

Consequently,

$$[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1]\mathbf{I}_s(\omega) = q\Omega \int_0^\infty f(r) r^3 dr \int_0^{2\pi} d\varphi' \int_0^\pi (\sin\theta' \cos\alpha \hat{\varphi}' + \sin\alpha \cos\theta' \hat{\varphi}'') e^{-i(\omega/c) r \cos\theta'} d(\cos\theta')$$
$$= q\Omega \cos\alpha \int_0^\infty f(r) r^3 dr \int_0^{2\pi} \hat{\varphi}' d\varphi' \int_0^\pi sin^2\theta' e^{-i(\omega/c) r \cos\theta'} d\theta'$$
$$+ \hat{\varphi}'' q\Omega \sin\alpha \int_0^\infty f(r) r^3 dr \int_0^{2\pi} d\varphi' \int_0^\pi \sin\theta' \cos\theta' e^{-i(\omega/c) r \cos\theta'} d\theta'$$
$$= \hat{\varphi}'' q\Omega \sin\alpha \int_0^\infty f(r) r^3 dr \int_0^{2\pi} d\varphi' \int_0^\pi \sin\theta' \cos\theta' e^{-i(\omega/c) r \cos\theta'} d\theta'$$

where in canceling the term containing $\hat{\varphi}'$ we have used the fact that

$$\int_0^{2\pi} \hat{\varphi}' d\varphi' = \hat{\mathbf{y}}' \int_0^{2\pi} \cos\varphi' d\varphi' - \hat{\mathbf{x}}' \int_0^{2\pi} \sin\varphi' d\varphi' = \mathbf{0}.$$

Moreover,

$$\int_0^\pi \sin\theta' \cos\theta' e^{-i(\omega/c) r \cos\theta'} d\theta' = \int_{-1}^{+1} x e^{-i\frac{\omega}{c} r x} dx$$
$$= \frac{2i}{(\omega r/c)^2} \left[\left(\frac{\omega}{c} r\right) \cos\left(\frac{\omega}{c} r\right) - \sin\left(\frac{\omega}{c} r\right) \right].$$

Hence,

$$[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1]\mathbf{I}_s(\omega) = \hat{\varphi}'' 4\pi i q\Omega \sin\alpha \int_0^\infty f(r) r^3 \frac{1}{\left(\frac{\omega}{c} r\right)^2} \times \left[\left(\frac{\omega}{c} r\right) \cos\left(\frac{\omega}{c} r\right) - \sin\left(\frac{\omega}{c} r\right) \right] dr. \quad \text{(A3)}$$

Upon substituting (A3) into (28) one obtains (43). QED

Proof of Eq. (29)

Given that the current is in general made up of two components, namely, $\mathbf{j}_l$ and $\mathbf{j}_s$, one would expect the power spectrum to be, in general, given by

$$P = P_l + P_s + \int_{S^2} d\hat{\mathbf{s}} \int_T (Q_l Q_s^* + Q_l^* Q_s) dt \quad \text{(A4)}$$

where the asterisk denotes the complex conjugate and where

$$Q_l \equiv \frac{1}{4\pi c^3 T} \sum_{n=0}^{\infty} [\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1] \mathbf{C}_{\ln}(\Gamma_{\ln}) \Gamma_{\ln} e^{-i\Gamma_{\ln} t}, \quad \text{(A5)}$$

and

$$Q_s \equiv \frac{1}{4\pi c^3 T} \sum_{n=0}^{\infty} [\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1] \mathbf{C}_{sn}(\Gamma_{sn}) \Gamma_{sn} e^{-i\Gamma_{sn} t}. \quad \text{(A6)}$$

Therefore in order to prove that P reduces to the mere sum of P_l and P_s we must show that the term that mixes $\mathbf{j}_l$ and $\mathbf{j}_s$ (or equivalently $\tilde{\mathbf{j}}_l$ and $\tilde{\mathbf{j}}_s$), i.e., $\int_{S^2} d\hat{\mathbf{s}} \int_T (Q_l Q_s^* + Q_l^* Q_s) dt$, vanishes. (In fact it will turn out that $Q_l Q_s^*$ and $Q_l^* Q_s$ vanish separately.)

We shall prove that the vanishing of the mixing term is due to the vanishing of

$$\Phi \equiv \int_0^{2\pi} d\varphi' \int_{-1}^{+1} \{[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1]\hat{\mathbf{z}}\} \cdot \{[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1](\hat{\mathbf{z}} \times \hat{\mathbf{r}})\} \times e^{-i(\omega/c) r \cos\theta'} d(\cos\theta') \sim Q_l Q_s^*. \quad \text{(A7)}$$

For $Q_l^* Q_s$ the same will apply; all one needs to do is to take the complex conjugate of Φ.

From Eq. (A2) we have

$$\{[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1]\hat{\mathbf{z}}\} \cdot \{[\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot) - 1](\hat{\mathbf{z}} \times \hat{\mathbf{r}})\}$$
$$= \{\cos\alpha\hat{\mathbf{s}} - \hat{\mathbf{z}}\} \cdot \{\cos\alpha \sin\theta' \hat{\varphi}' + \sin\alpha\cos\theta' \hat{\varphi}''\}$$
$$= -\cos\alpha\, \theta' \hat{\mathbf{z}} \cdot \hat{\varphi}', \tag{A8}$$

where only one term of the expansion has survived because the definitions of $\hat{\mathbf{s}}, \hat{\mathbf{z}}, \hat{\varphi}'$, and $\hat{\varphi}''$ impose that $\hat{\mathbf{s}}\cdot\hat{\varphi}' = \hat{\mathbf{s}}\cdot\hat{\varphi}'' = \hat{\mathbf{z}}\cdot\hat{\varphi}'' = 0$.

In order to evaluate $\hat{\mathbf{z}}\cdot\hat{\varphi}'$ we need to express $\hat{\mathbf{z}}$ in the new rotated coordinate system. A common representation of a three-dimensional rotation about an arbitrary axis labeled by a unit vector $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ is the matrix [21]

$$\mathfrak{R} = \begin{pmatrix} tn_x^2 + c & tn_x n_y - sn_z & tn_x n_z + sn_y \\ tn_x n_y + sn_z & tn_y^2 + c & tn_y n_z - sn_x \\ tn_x n_z - sn_y & tn_y n_z + sn_x & tn_z^2 + c \end{pmatrix}$$

where s, c, and t are defined in terms of the rotation angle γ as $s \equiv \sin\gamma, c \equiv \cos\gamma$, and $t \equiv 1 - \cos\gamma$. In our case $\gamma = \alpha$. We now need to determine the components of $\hat{\mathbf{n}}$ in the old (unprimed) coordinate system. Because the axis of rotation is perpendicular to the plane formed by $\hat{\mathbf{s}}$ and $\hat{\mathbf{z}}$ we have $\hat{\mathbf{n}}\cdot\hat{\mathbf{s}} = 0$. In addition to the requirement that $\hat{\mathbf{n}}$ be a unit vector and the fact that the axis of rotation passes through the origin in our case, this orthogonality of $\hat{\mathbf{n}}$ and $\hat{\mathbf{s}}$ allows us to determine the components of $\hat{\mathbf{n}}$ in the old coordinate system. A trivial calculation shows that $\hat{\mathbf{n}} = (\pm\sin\beta, \mp\cos\beta, 0)$. We arbitrarily choose $\hat{\mathbf{n}} = (\sin\beta, -\cos\beta, 0)$. Thus the Cartesian coordinates of $\hat{\mathbf{z}}$ in the new (primed) coordinate system are

$$\mathfrak{R}\hat{\mathbf{z}} = \mathfrak{R}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\alpha\cos\beta \\ -\sin\alpha\sin\beta \\ \cos\alpha \end{pmatrix}.$$

Therefore

$$\hat{\mathbf{z}}\cdot\hat{\varphi}' = \sin\alpha\sin(\varphi' - \beta). \tag{A9}$$

Substituting (A9) into (A8) and then injecting the result into (A7) yields

$$\Phi = -\sin\alpha\cos\alpha \int_0^{2\pi} d\varphi' \sin(\varphi' - \beta) \times \int_0^{\pi} \sin^2\theta' e^{\pm i(\omega/c) r \cos\theta'} d\theta'. \tag{A10}$$

But

$$\int_0^{2\pi} d\varphi' \sin(\varphi' - \beta) = 0, \tag{A11}$$

therefore

$$\Phi = 0 \Rightarrow Q_l Q_s^* = 0 = Q_l^* Q_s.$$

QED

[1] S. Coleman, in *Electromagnetism: Paths to Research*, edited by D. Teplitz (Plenum, New York, 1982); P. Pearle, in *Electromagnetism: Paths to Research*, edited by D. Teplitz (Plenum, New York, 1982).

[2] A. D. Yaghjian, *Relativistic Dynamics of a Charged Sphere*, Lecture Notes in Physics, Vol. 11 (Springer-Verlag, Berlin, 1992).

[3] F. Rohrlich, Am. J. Phys. **65**, 1051 (1997).

[4] H. Spohn, *Dynamics of Charged Particles and their Radiation Field* (Cambridge University Press, Cambridge, U.K., 2004).

[5] G. W. Ford and R. F. O'Connell, Phys. Lett. A **157**, 217 (1991).

[6] F. Rohrlich, Phys. Lett. A **303**, 307 (2002).

[7] G. W. Ford and R. F. O'Connell, Phys. Rev. A **44**, 6386 (1991); Phys. Lett. A **158**, 31 (1991).

[8] C.-S. Ng, Phys. Rev. E **47**, 2038 (1993).

[9] R. Lieu, W. I. Axford, and J. F. McKenzie, Phys. Rev. E **55**, 1872 (1997).

[10] K. H. Mariwalla and N. D. Hari Dass, IMSC Report No. 98/01/03; e-print physics/0205046.

[11] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1998).

[12] E. A. Marengo and A. J. Devaney, J. Math. Phys. **39**, 3643 (1998).

[13] T. B. Hansen and A. D. Yaghjian, *Plane-Wave Theory of Time-Domain Fields: Near-Field Scanning Applications* (IEEE Press, New York, 1999).

[14] E. Heyman, J. Math. Phys. **37**, 658 (1996).

[15] J. G. Proakis and D. G. Manolakis, *Introduction to Digital Signal Processing* (Maxwell Macmillan, New York, 1988).

[16] R. Lehnert and R. Potting, Phys. Rev. D **70**, 125010 (2004).

[17] I. Dymnikova, A. Sakharov, J. Ulbricht, and J. Zhao, e-print hep-ph/0111302.

[18] D. Bohm and M. Weinstein, Phys. Rev. **74**, 1789 (1948).

[19] X. Ming, H. Huang, C. A. Zorman, M. Mehregany, and M. L. Roukes, Nature (London) **421**, 496 (2003).

[20] P. J. Burke, S. Li, and Z. Yu, e-print cond-mat/0408418.

[21] L. A. Pars, *A Treatise on Analytical Dynamics* (Wiley, New York, 1965).